Construction of a Lyapunov Function for 1-D Linear Hyperbolic 2×2 Systems

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Consider a linear not strictly hyperbolic 2×2 system

$$\begin{aligned} \partial_t x_1(z,t) &= a(z)\partial_z x_1(z,t) + b_{11}(z)x_1(z,t) + b_{12}(z)x_2(z,t), \\ \partial_t x_2(z,t) &= a(z)\partial_z x_2(z,t) + b_{21}(z)x_1(z,t) + b_{22}(z)x_2(z,t) \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{pmatrix} x_1(\ell,t) \\ x_2(\ell,t) \end{pmatrix} = \Lambda \begin{pmatrix} x_1(0,t) \\ x_2(0,t) \end{pmatrix}. \tag{2}$$

For the system (1)—(2), we consider the Cauchy problem with initial conditions $(x_1(z,0), x_2(z,0)) = (x_{10}(z), x_{20}(z))$. We assume, that $a \in C^1([0,\ell],\mathbb{R})$, $b_{ij} \in C^1([0,\ell],\mathbb{R})$, $i,j = 1,2$, $i \neq j$, $a(z) > 0$ for all $z \in [0,\ell]$, and $\Lambda = (\lambda_{ij})_{i,j=1,2}$ is a constant matrix

A Lyapunov function of the following structure

$$V(x_1,x_2) = \int_0^\ell (q_1(z)x_1^2(z) + q_2(z)x_2^2(z))\,dz, \quad i,j = 1,2, \quad i \neq j,$$

where $q_i \in C^1([0,\ell],\mathbb{R}_{>0})$ is called the diagonal Lyapunov function [1]. We establish necessary conditions for the existence of the diagonal Lyapunov function.

Theorem 1 *Let for the linear system* (1)—(2) *there exists the diagonal Lyapunov function, then the following inequalities hold*

$$|\lambda_{11}| < \exp\Big(-\int_0^\ell \frac{b_{11}(z)}{a(z)}\,dz\Big), \quad |\lambda_{22}| < \exp\Big(-\int_0^\ell \frac{b_{22}(z)}{a(z)}\,dz\Big),$$

$$|\lambda_{12}||\lambda_{21}| < \exp\Big(\int_0^\ell \frac{b_{11}(z) + b_{22}(z) - 2\sqrt{(b_{12}(z)b_{21}(z))_+}}{a(z)}dz\Big),$$

$$\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22} < \frac{1}{2}\exp\Big(2\int_0^\ell \frac{b_{11}(z) + b_{22}(z) - 2\sqrt{(b_{12}(z)b_{21}(z))_+}}{a(z)}dz\Big).$$

Here $(f(z))_+ = \begin{cases} f(z), & f(z) \geq 0, \\ 0, & f(z) < 0. \end{cases}$

A non-diagonal Lyapunov function of the following form

$$V(x) = \int_0^\ell e^{\mu z} x^{\mathrm{T}}(z)P(z)x(z)\,dz, \quad P(z) = \frac{1}{a(z)}\Omega_0^z P_0 (\Omega_0^z)^{\mathrm{T}}$$

is proposed, where $\mu > 0$, Ω_0^z is the solution of the following linear matrix differential equation

$$\frac{d\Omega_0^z}{dz} = \frac{1}{a(z)} B^{\mathrm{T}}(z)\Omega_0^z, \quad \Omega_0^0 = \mathrm{Id}, \quad B(z) = (b_{ij}(z))_{i,j=1,2},$$

P_0 is a positive definite matrix. This Lyapunov function leads to the following conditions for the asymptotic stability of the system (1)—(2). We denote $r_\sigma(.)$ is a spectral radius of a corresponding matrix.

Theorem 2 *Let* $r_\sigma((\Omega_0^\ell)^{\mathrm{T}}\Lambda) < 1$*, then the linear system* (1)—(2) *is exponentially stable.*

Since the explicit form of matrices Ω_0^z is usually unknown, we proposed the non-diagonal Lyapunov function with the kernel

$$P(z) = \frac{1}{a(z)} W^{\mathrm{T}}(z) P_0 W(z),$$

P_0 is a positive definite matrix, $W(z) = \exp U(z)$, where $U(z)$ is the partial sum of the Magnus series [2]

$$\begin{aligned} U(z) = & \int_0^z C(\tau)\, d\tau - \frac{1}{2}\int_0^z \left[C(\tau), \int_0^\tau C(\sigma)\, d\sigma \right] d\tau \\ & + \frac{1}{4}\int_0^z \left[C(\tau), \int_0^\tau \left[C(\sigma), \int_0^\sigma C(\rho)\, d\rho \right] d\sigma \right] d\tau \\ & + \frac{1}{12}\int_0^z \left[\left[C(\tau), \int_0^\tau C(\sigma)\, d\sigma \right], \int_0^\tau C(\sigma)\, d\sigma \right] d\tau. \end{aligned}$$

Based on this Lyapunov function sufficient conditions of the asymptotic stability of the system (1)—(2) are obtained. We have compared various exponential stability conditions obtained using the diagonal and non-diagonal Lyapunov functions.

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